

# Exact eigenspectrum of the symmetric simple exclusion process on the complete, complete bipartite, and related graphs

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We cast the infinitesimal generator of the symmetric simple exclusion process on a graph as a quantum spin- $\frac{1}{2}$  ferromagnetic Heisenberg model and show that its eigenspectrum can be obtained by elementary techniques on the complete, complete bipartite, and related multipartite graphs. Some of the resulting infinitesimal generators are formally identical to homogeneous as well as mixed higher spins models. The degeneracies of the eigenspectra are described in detail, and we give a neat derivation of the outer multiplicities appearing in the Clebsch-Gordan series for arbitrary spin- $s$  representations of the  $SU(2)$  not easily found elsewhere. We mention in passing how our results fit within the related questions of a ferromagnetic ordering of energy levels and a conjecture according to which the spectral gaps of the random walk and the interchange process on finite simple graphs must be equal.

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## I. INTRODUCTION

Exclusion processes, together with the contact process and the Glauber-Ising model, are one of the most fundamental models in the field of nonequilibrium interacting particle systems [1]. In physics, exclusion processes are the simplest models that furnish nontrivial results on a number of basic issues, such as the relaxation dynamics of an interacting gas towards the thermodynamic equilibrium or the dynamics of shock waves in discrete models for inviscid fluids [2]. In the one-dimensional linear chain, simple exclusion processes, either symmetric or asymmetric, under periodic or more general open boundary conditions, have been analyzed and their relationship with other models of interest spanned a wealth of mathematical physics during the last two decades [3–9].

The investigation of exclusion processes on general graphs, however, has been timid in the physics literature, except perhaps for the study of some multi-lane traffic models that include junctions and bifurcations [10], and most studies do not go beyond the level of simple mean field analyses or numerical simulations. In the mathematical literature, otherwise, the study of random walks and exclusion processes on graphs is a relatively hot topic connected with deep results in probability, group theory, harmonic analysis, and combinatorics [11–13]. Unfortunately, this mathematical literature is inaccessible to most practicing physicists, even the well prepared ones, with possibly useful results buried beneath loads of advanced prerequisites, hardcore formalism, and subtle rationale.

The objective of this article is to show that the infinitesimal generator of the symmetric simple exclusion process (SSEP) on the complete, complete bipartite, and closely related graphs can be solved by elementary techniques that belong in the toolbox of every trained physicist. We believe that the explicit calculations presented here simplify the understanding of the models and also open some interesting perspectives.

## II. THE SSEP ON A GRAPH

Let  $G = (V, E)$  be a finite simple (without loops) undirected connected graph of order  $N$  with vertex set  $V = \{1, \dots, N\}$  and edge set  $E \subseteq V \times V$ . To each  $i \in V$  we attach a random variable  $\sigma_i$  taking values in  $\{-1, +1\}$ . If  $\sigma_i = -1$  we say that vertex  $i$  is empty and if  $\sigma_i = +1$  we say that vertex  $i$  is occupied by a particle. The state of the system is specified by the configuration  $\sigma = (\sigma_1, \dots, \sigma_N)$  in  $\Omega = \{-1, +1\}^V$ . The SSEP( $G$ ) is the continuous-time Markov jump process that describes the transitions of a set of  $n$  itinerant particles,  $1 \leq n \leq N$ , between the connected vertices of  $G$ . In the SSEP( $G$ ), each particle chooses, sequentially and at exponentially distributed times, one of its adjacent vertices to jump to provided the target vertex is empty, otherwise the jump attempt fails and the process continues. Clearly, when  $n = 1$  we have the simple random walk on  $G$ . When  $n \geq 2$ , exclusion between particles comes into play and the process becomes more interesting.

We introduce vector spaces in the description of the SSEP( $G$ ) by turning  $\Omega$  into  $(\mathbb{C}^2)^{\otimes V}$ ,  $\sigma$  into  $|\sigma\rangle = |\sigma_1\rangle \otimes \dots \otimes |\sigma_N\rangle$ , and setting  $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  to identify respectively an empty and an occupied vertex. A little reflection shows that within this vector space scenario the infinitesimal generator of the time evolution of the SSEP( $G$ ) can be written as (for detailed derivations see, e.g., [7–9])

$$\mathcal{H} = \sum_{i \sim j} (1 - \mathcal{P}_{ij}), \quad (1)$$

where  $i \sim j$  stands for pairs of connected vertices of  $G$  and  $\mathcal{P}_{ij}$  is the operator that transposes the states of vertices  $i$  and  $j$ ,

$$\mathcal{P}_{ij} |\dots, \sigma_i, \dots, \sigma_j, \dots\rangle = |\dots, \sigma_j, \dots, \sigma_i, \dots\rangle. \quad (2)$$

As is well known,  $\mathcal{P}_{ij}$  can be written in terms of Pauli spin matrices as

$$\mathcal{P}_{ij} = \frac{1}{2}(1 + \vec{\sigma}_i \cdot \vec{\sigma}_j) = \frac{1}{2}(1 + \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z). \quad (3)$$

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Inserting this  $\mathcal{P}_{ij}$  in Eq. (1) gives

$$\mathcal{H} = \frac{1}{2} \sum_{i \sim j} (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j). \quad (4)$$

We then see that  $\mathcal{H}$  is, to within a diagonal term, exactly the Hamiltonian of the isotropic Heisenberg spin- $\frac{1}{2}$  quantum ferromagnet over  $G$  [14]. The ground states of  $\mathcal{H}$  have eigenvalue zero and correspond (under a probabilistic normalization) to the stationary states of the process.

Operator (4) is positive semi-definite and the master equation governing the time evolution of the probability density  $P(\sigma, t)$  of observing configuration  $\sigma$  at instant  $t$  reads  $\partial_t P(\sigma, t) = -\mathcal{H}P(\sigma, t)$ . One is usually interested in the spectral gap of  $\mathcal{H}$ , which is the inverse of the leading characteristic time scale of the process related with the time it takes to approach the stationary state. Conservation of particles in the SSEP( $G$ ) implies that  $\mathcal{H}$  commutes with the total number of particles operator

$$\mathcal{N} = \frac{1}{2} \sum_{i=1}^N (1 + \sigma_i^z) = \frac{N}{2} + S^z, \quad (5)$$

where  $S^z$  is the  $z$ -axis “polarization” operator. It follows that  $\mathcal{H}$  is block-diagonal,  $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ , with each block  $\mathcal{H}_n$  acting on its respective invariant subspace  $\Omega_n$  of dimension  $\dim \Omega_n = \binom{N}{n} = N!/n!(N-n)!$ . The eigenspectrum of  $\mathcal{H}$  is also symmetric about  $n = N/2$ , because it commutes with the “spin flip” operator

$$\mathcal{U} = \prod_{i=1}^N \sigma_i^x, \quad (6)$$

that transforms particles into holes and vice-versa,  $\mathcal{U}|\sigma_1, \dots, \sigma_N\rangle = |-\sigma_1, \dots, -\sigma_N\rangle$ , taking a state with  $n$  particles into a state with  $N-n$  particles. The eigenspectra in the sectors of  $n$  and  $N-n$  particles are thus identical.

In what follows we investigate operator (4) on a couple of different graphs and show that some of them can be analyzed by elementary techniques.

### III. THE SSEP ON THE COMPLETE GRAPH

In the complete graph  $K_N$ , every pair of distinct vertices is connected by a unique edge; see Fig. 1. For this graph, the infinitesimal generator (4) reads

$$\mathcal{H} = \frac{1}{2} \sum_{1 \leq i < j \leq N} (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j). \quad (7)$$

Rearranging the summation in Eq. (7) and noticing that  $(\vec{\sigma}_i)^2 = 3_i$  we eventually arrive at

$$\mathcal{H} = \frac{N}{2} \left( \frac{N}{2} + 1 \right) - \left( \frac{1}{2} \sum_{i=1}^N \vec{\sigma}_i \right)^2. \quad (8)$$

This  $\mathcal{H}$  is but the Curie-Weiss version of the spin- $\frac{1}{2}$  ferromagnetic Heisenberg model without the overall multiplicative

$1/N$  term usually included to keep the energy per spin an intensive quantity, since we are not doing any thermodynamics here [15]. In the basis simultaneously diagonal in the total spin squared operator

$$\vec{S}^2 = \left( \frac{1}{2} \sum_{i=1}^N \vec{\sigma}_i \right)^2 \quad (9)$$

with eigenvalues  $S(S+1)$ ,  $S = S_{\min}, S_{\min} + 1, \dots, N/2$ , where  $S_{\min} = 0$  or  $1/2$  depending whether  $N$  is even or odd, and in the total  $z$ -axis component  $S^z$  defined in Eq. (5) with eigenvalues  $M = -S, -S+1, \dots, +S$ , the eigenvalues of  $\mathcal{H}$  read

$$E_N(S, M) = \frac{N}{2} \left( \frac{N}{2} + 1 \right) - S(S+1), \quad (10)$$

i.e.,  $E_N(S, M) = E_N(S)$ , independent on  $M$ . The degeneracy of  $E_N(S)$  is given by  $g_N(S) = (2S+1) \times d_{1/2}(N, S)$ , where the factor  $2S+1$  comes from the degeneracy in the  $S^z$  values of rotationally invariant operators like  $\mathcal{H}$ , and the  $d_{1/2}(N, S)$  comes from the fact that there exists many possible combinations of the  $N$  elementary spins summing up to a definite value of  $S$ . This last factor is given by the outer multiplicity of the irreducible representation  $\mathcal{D}^{(S)}$  appearing in the Clebsch-Gordan series

$$[\mathcal{D}^{(1/2)}]^{\otimes N} = \bigoplus_{S=S_{\min}}^{N/2} d_{1/2}(N, S) \mathcal{D}^{(S)}, \quad (11)$$

and can be shown to be given by (cf. appendix)

$$d_{1/2}(N, S) = \binom{N}{\frac{1}{2}N+S} - \binom{N}{\frac{1}{2}N+S+1}. \quad (12)$$

We see from Eqs. (10) and (12) that  $E_N(S = N/2) = 0$  with a  $(N+1)$ -fold degeneracy. These values have a simple interpretation: the SSEP( $G$ ) has a zero eigenvalue on each of its  $N+1$  sectors of total particle number  $n = N/2 + M = 0, 1, \dots, N$ . That the stationary states of  $\mathcal{H}$  occur in the sectors of  $S = N/2$  is just another statement of the well known fact that the ground states of ferromagnetic Heisenberg models have maximum possible total  $S$ . The right eigenvectors corresponding to the zero eigenvalues are the stationary states of the process, explicitly given by

$$|\Phi_0^N(n)\rangle = \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} |1_{i_1}, 1_{i_2}, \dots, 1_{i_n}\rangle; \quad (13)$$

notice the probabilistic normalization of  $|\Phi_0^N(n)\rangle$ , not the quantum-mechanical one. The summation in Eq. (13) runs over all combinations of the  $n$  particle positions  $i_1, i_2, \dots, i_n$  among the  $N$  available vertices of the graph.

For processes that conserve the total number of particles like the SSEP( $G$ ), a basis diagonal in  $n$  is more useful. In the  $|S, M\rangle$  basis, each invariant subspace  $\Omega_n$  of fixed  $n = 0, 1, \dots, N$  is spanned by the states with  $M = -N/2 + n$  fixed and  $|-N/2 + n| \leq S \leq N/2$ , with the given  $|S, M = -N/2 + n\rangle$  states within  $\Omega_n$  bearing their original

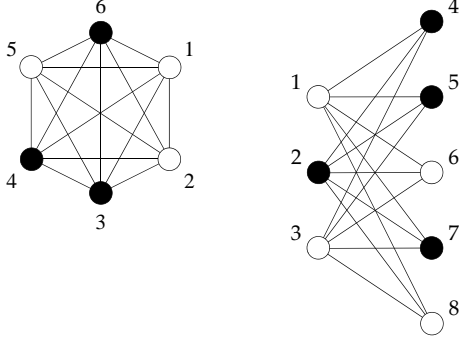


FIG. 1. Left: Complete graph  $K_6$  with  $n = 3$  particles (black circles) occupying some of its vertices. Right: Complete bipartite graph  $K_{3,5}$  occupied by  $n = 4$  particles.

multiplicity  $d_{1/2}(N, S)$ . This completely characterizes the eigenspectrum of  $\mathcal{H}$  in each of its invariant subspaces.

Tables I and II illustrate the SSEP( $K_N$ ) in the concrete case of  $N = 8$ . The eigenvalues of SSEP( $K_8$ ) and their degeneracies appear in Tab. I. The eigenspectrum in terms of the total number of particles appears in Tab. II and is clearly symmetric about  $n = N/2$ , as we anticipated in Sec. II.

The spectral gap  $\Delta_N$  of  $\mathcal{H}$  is given by the smallest nonzero eigenvalue of  $\mathcal{H}$ , and is related with the characteristic time  $\tau_N$  it takes for the process to exponentially decay to its stationary state by  $\tau_N^{-1} = \Delta_N$ . For the SSEP( $K_N$ ),  $\Delta(K_N) = E_N(N/2 - 1) = N$  is the same in every invariant sector of constant particle number of the process, except in the one-dimensional sectors of  $n = 0$  and  $n = N$ , for which there is no gap at all. That  $\Delta(K_N) = N$  hints at the fact that the characteristic time  $\tau_N$  scales with the system size as  $N\tau_N = N\Delta(K_N)^{-1} = 1$ , i.e., the interacting particle system relaxes to its stationary state after just one step, irrespective of  $N$ . This has to do with the fact that on  $K_N$  any vertex can be reached from any other one through a single jump. We also see that the energy levels are monotone decreasing in  $S$ ,  $E_N(S) > E_N(S')$  whenever  $S < S'$ , a property that has been dubbed “ferromagnetic ordering of energy levels” for ferromagnetic SU(2)-invariant quantum spin models [16]. There are both examples and counterexamples of models holding this property [17]. These results are akin to the so-called Aldous’ spectral gap conjecture (ca. 1990), according to which the gap of the single particle random walk ( $n = 1$ ) should be equal to the gap of the interchange process ( $n = N$ ) on any finite simple graph [18]. This conjecture spawned some original results in probability and mathematical physics, mostly on the last decade, and was proved in the most general form only recently through a *mélange* of group-theoretical, probabilistic, and combinatorial arguments [19].

#### IV. THE SSEP ON THE COMPLETE BIPARTITE GRAPH

The complete bipartite graph  $K_{N_1, N_2}$  is the simple undirected graph with partitioned vertex set  $V = V_1 \cup V_2$  with  $|V_i| = N_i$ ,  $i = 1, 2$ , and  $V_1 \cap V_2 = \emptyset$  such that every vertex in  $V_1$  is connected to every vertex in  $V_2$  by a unique edge; see Fig. 1.

TABLE I. Eigenspectrum of the SSEP( $G$ ) on the complete graph  $K_8$ . The degeneracies  $g_N(S)$  are given as  $(2S + 1) \times d_{1/2}(N, S)$ . Notice that  $\sum_S g_N(S) = 2^8$ , as it should be.

$S$	0	1	2	3	4
$E_N(S)$	20	18	14	8	0
$g_N(S)$	$1 \times 14$	$3 \times 28$	$5 \times 20$	$7 \times 7$	$9 \times 1$

TABLE II. Characterization of the invariant subspaces  $\Omega_n$  of the SSEP( $K_8$ ). The multiplicities of the  $|S, M = -N/2 + n\rangle$  states within each  $\Omega_n$  are given in the last column as  $(S^{d_{1/2}(N, S)})$ .

$n = \frac{1}{2}N + M$	$\dim \Omega_n$	$(S^{d_{1/2}(N, S)})$
0	1	$(4^1)$
1	8	$(3^7)(4^1)$
2	28	$(2^{20})(3^7)(4^1)$
3	56	$(1^{28})(2^{20})(3^7)(4^1)$
4	70	$(0^{14})(1^{28})(2^{20})(3^7)(4^1)$
5	56	$(1^{28})(2^{20})(3^7)(4^1)$
6	28	$(2^{20})(3^7)(4^1)$
7	8	$(3^7)(4^1)$
8	1	$(4^1)$

For this graph,

$$\mathcal{H} = \frac{1}{2} \sum_{i_1 \in V_1} \sum_{i_2 \in V_2} (1 - \vec{\sigma}_{i_1} \cdot \vec{\sigma}_{i_2}) = \frac{1}{2} N_1 N_2 - 2 \vec{S}_1 \cdot \vec{S}_2, \quad (14)$$

where the operators  $\vec{S}_1$  and  $\vec{S}_2$  are given by

$$\vec{S}_1 = \frac{1}{2} \sum_{i_1 \in V_1} \vec{\sigma}_{i_1}, \quad \vec{S}_2 = \frac{1}{2} \sum_{i_2 \in V_2} \vec{\sigma}_{i_2}. \quad (15)$$

It is a matter of simple algebra to demonstrate that the  $\vec{S}_i$ ,  $i = 1, 2$ , obey  $\vec{S}_i \times \vec{S}_i = i\vec{S}_i$ , being thus legitimate spin operators. The magnitude of the spin  $\vec{S}_i$  is  $S_i = N_i/2$ .

Notice that in representing the occupation state of  $V_i$  by a state of  $\vec{S}_i$  indexed by the value of its  $S_i^z$  component through the relation  $n_i = N_i/2 + m_i$ ,  $m_i = -N_i/2, -N_i/2 + 1, \dots, +N_i/2$ , we have promoted a reduction of the dimension of the configuration space associated with  $V_i$  from  $2^{N_i}$  to  $N_i + 1$ . This reduction comes from lumping equivalent configurations obtained by permutations of the particles among the vertices of  $V_i$  into a single representative state. The result is that the  $2^N$ -dimensional original problem can be treated as a  $(N_1 + 1)(N_2 + 1)$ -dimensional problem as far as the determination of the eigenspectrum is concerned. If the eigenstates of (14) become needed, e.g., to calculate correlation functions or block entropies, one must reconstruct them from the original  $2^N$  states by appropriate combinations of permutations.

In terms of the total spin  $\vec{S} = \vec{S}_1 + \vec{S}_2$ , we have  $2\vec{S}_1 \cdot \vec{S}_2 = \vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2$ . In the basis diagonal in the complete set of commuting operators  $\vec{S}_1^2, \vec{S}_2^2, \vec{S}^2$ , and  $S^z = S_1^z + S_2^z$ , the eigenvalues

of (14) are given by

$$E_{N_1, N_2}(S_1, S_2, S, M) = \frac{1}{2}N_1N_2 + S_1(S_1 + 1) + S_2(S_2 + 1) - S(S + 1), \quad (16)$$

or, in more compact form, by

$$E_N(S) = (N/2 - S)(N/2 + S + 1), \quad (17)$$

with  $|S_1 - S_2| \leq S \leq S_1 + S_2$  and  $|M| \leq S$ . The number of particles in the system is given by  $n = N/2 + M$ , as before.

For each of the  $\min\{2S_1 + 1, 2S_2 + 1\}$  values of  $S$ ,  $E_N(S)$  is  $2S + 1$  degenerate due to its independence on  $M$ . Overall,  $M$  is in the range  $-|S_1 - S_2| \leq M \leq S_1 + S_2$ , and for any given  $M$  we have  $\max\{|M|, |S_1 - S_2|\} \leq S \leq S_1 + S_2$ . The lowest eigenvalue of (14) lies in the sector of maximum  $S = S_1 + S_2 = N_1/2 + N_2/2 = N/2$ —as expected for a “ferromagnetic” model—, with a  $N + 1$  degeneracy ( $M = -N/2, -N/2 + 1, \dots, +N/2$ ) associated with the  $N + 1$  stationary states of the process, one within each invariant sector of constant number of particles ( $n = N/2 + M = 0, 1, \dots, N$ ). The steady states are given by the same  $|\Phi_0^N(n)\rangle$  as in Eq. (13).

The spectral gap of the process is given by  $\Delta(K_{N_1, N_2}) = E_N(N/2 - 1) = N + 2$ , and like the gap of the SSEP( $K_N$ ) is the same in every invariant sector of constant particle number. It is also clear from Eq. (16) or Eq. (17) that the energy levels observe the ferromagnetic ordering property mentioned in Sec. III, namely,  $E_N(S) > E_N(S')$  whenever  $S < S'$ , providing yet another example of such systems [16, 17].

## V. THE SSEP ON MULTIPARTITE GRAPHS

The cases analyzed so far lead naturally to the SSEP on generalized multipartite graphs. In particular, two types of multipartite graphs are of interest: complete multipartite graphs and concatenated (chained) bipartite graphs. Although the SSEP on this second type of graphs cannot be solved by elementary techniques—actually, some of them cannot be exactly solved at all—, they give rise to infinitesimal generators that may appeal in other modeling circumstances.

### A. The complete multipartite graph

The complete multipartite graph  $K_{Q_1, \dots, Q_N}$  is the simple undirected graph with partitioned vertex set  $V = V_1 \cup \dots \cup V_N$  with  $|V_i| = Q_i$ ,  $i = 1, \dots, N$ , and  $V_i \cap V_j = \emptyset$  for  $i \neq j$  such that every two vertices from different sets  $V_i$  and  $V_j$  are adjacent. When  $Q_1 = \dots = Q_N = Q$ , we have the  $Q$ -regular complete multipartite graph  $K_Q^N$ ; see Fig. 2.

Following our previous approach, we associate to each disjoint subset  $V_k$  a spin- $Q_k/2$  operator  $\vec{S}_k$  acting on its own subspace of dimension  $Q_k + 1$  given by

$$\vec{S}_k = \frac{1}{2} \sum_{i_k \in V_k} \vec{\sigma}_{i_k}. \quad (18)$$

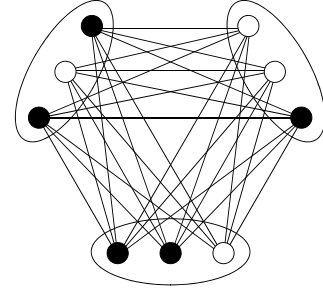


FIG. 2. The 3-regular complete multipartite graph  $K_{3,3,3} = K_3^3$  occupied by  $n = 5$  particles (black circles). Clearly,  $K_{1, \dots, 1} = K_1^N = K_N$ , the complete graph considered in section III.

In terms of these spin operators, the infinitesimal generator of the SSEP on the complete multipartite graph  $K_{Q_1, \dots, Q_N}$  reads

$$\mathcal{H} = \frac{1}{4} \left( \sum_{i=1}^N Q_i \right)^2 - \frac{1}{4} \sum_{i=1}^N Q_i^2 - \left( \sum_{i=1}^N \vec{S}_i \right)^2 + \sum_{i=1}^N \vec{S}_i^2. \quad (19)$$

On the  $Q$ -regular complete multipartite graph  $K_Q^N$ , all  $\vec{S}_i$  are equivalent spin- $Q/2$  operators. In this case, taking into account that  $\vec{S}_i^2 = \frac{1}{2}Q(\frac{1}{2}Q + 1)$ , the infinitesimal generator (19) of the SSEP( $K_Q^N$ ) becomes

$$\mathcal{H} = \frac{1}{2}NQ\left(\frac{1}{2}NQ + 1\right) - \left( \sum_{i=1}^N \vec{S}_i \right)^2. \quad (20)$$

This operator is formally identical with the Hamiltonian of a quantum spin- $Q/2$  Curie-Weiss model and can be analyzed along the same lines as the spin- $\frac{1}{2}$  operator (12) in Sec. III. If we define a total spin operator  $\vec{S} = \sum_i \vec{S}_i$ , then the eigenvalues of (20) can be read off immediately as

$$E_Q^N(S, M) = \frac{1}{2}NQ\left(\frac{1}{2}NQ + 1\right) - S(S + 1), \quad (21)$$

with  $S = S_{\min}, S_{\min} + 1, \dots, NQ/2$  and  $M = -S, -S + 1, \dots, +S$ , where  $S_{\min} = 1/2$  if  $Q$  and  $N$  are both odd and  $S_{\min} = 0$  otherwise. Within each sector of fixed number of particles  $n = NQ/2 + M$ , the values of  $S$  range in the interval  $|M| \leq S \leq NQ/2$ . The degeneracies associated with the eigenvalues (21) are given by  $g_N(S) = (2S + 1) \times d_{Q/2}(N, S)$ , where now the outer multiplicities  $d_{Q/2}(N, S)$  determining the degeneracies in the  $S$  values are given by (cf. appendix)

$$d_{Q/2}(N, S) = b_{Q/2}(N, S) - b_{Q/2}(N, S + 1), \quad (22)$$

where the coefficients  $b_{Q/2}(N, M)$  are given by

$$b_{Q/2}(N, M) = \sum_{k \geq 0} (-1)^k \binom{N}{k} \binom{(\frac{1}{2}Q + 1)N + M - (Q + 1)k - 1}{\frac{1}{2}QN + M - (Q + 1)k}, \quad (23)$$

where the summation runs over  $k$  as long as the summing terms are non-null. Equations (22)–(23) are analogous to



TABLE III. Characterization of the invariant subspaces  $\Omega_n$  for the SSEP( $K_3^4$ ). The dimensionality of  $\Omega_n$  is given by the coefficient  $b_{Q/2}(N, M = \lfloor -NQ/2 + n \rfloor)$  given in Eq. (23). The multiplicities of the  $|S, M = -NQ/2 + n\rangle$  states within each  $\Omega_n$  are given in the last column as  $(S^{d_{Q/2}(N, S)})$ . The additional multiplicities coming from the permutation equivalent states within each set  $V_{Q_i}$ ,  $i = 1, \dots, N$ , are not accounted for in this table.

$n = \frac{1}{2}NQ + M$	$\dim \Omega_n$	$(S^{d_{Q/2}(N, S)})$
0	1	(6 <sup>1</sup> )
1	4	(5 <sup>3</sup> )(6 <sup>1</sup> )
2	10	(4 <sup>6</sup> )(5 <sup>3</sup> )(6 <sup>1</sup> )
3	20	(3 <sup>10</sup> )(4 <sup>6</sup> )(5 <sup>3</sup> )(6 <sup>1</sup> )
4	31	(2 <sup>11</sup> )(3 <sup>10</sup> )(4 <sup>6</sup> )(5 <sup>3</sup> )(6 <sup>1</sup> )
5	40	(1 <sup>9</sup> )(2 <sup>11</sup> )(3 <sup>10</sup> )(4 <sup>6</sup> )(5 <sup>3</sup> )(6 <sup>1</sup> )
6	44	(0 <sup>4</sup> )(1 <sup>9</sup> )(2 <sup>11</sup> )(3 <sup>10</sup> )(4 <sup>6</sup> )(5 <sup>3</sup> )(6 <sup>1</sup> )
7	40	(1 <sup>9</sup> )(2 <sup>11</sup> )(3 <sup>10</sup> )(4 <sup>6</sup> )(5 <sup>3</sup> )(6 <sup>1</sup> )
8	31	(2 <sup>11</sup> )(3 <sup>10</sup> )(4 <sup>6</sup> )(5 <sup>3</sup> )(6 <sup>1</sup> )
9	20	(3 <sup>10</sup> )(4 <sup>6</sup> )(5 <sup>3</sup> )(6 <sup>1</sup> )
10	10	(4 <sup>6</sup> )(5 <sup>3</sup> )(6 <sup>1</sup> )
11	4	(5 <sup>3</sup> )(6 <sup>1</sup> )
12	1	(6 <sup>1</sup> )

Eq. (12) and, indeed, they reduce to it when  $Q = 1$ . Coefficient (23) also corresponds to the dimension of the invariant subspace  $\Omega_M$  of fixed  $M = \lfloor -NQ/2 + n \rfloor$ ; notice that  $\dim \Omega_M = \dim \Omega_{-M}$ .

The same observations made for the eigenspectrum (10) hold here. The zero eigenvalue in Eq. 21 occurs in the sector of  $S = NQ/2$  with a  $NQ + 1$ -fold degeneracy, corresponding to the  $NQ + 1$  stationary states of the process, one within each subspace of constant particle number. The spectral gap  $\Delta(K_Q^N) = NQ$  of the process is also the same within each invariant sector of constant particle number. Finally, it is clear from Eq. (21) that the eigenvalues observe the ferromagnetic ordering  $E_Q^N(S) > E_Q^N(S')$  if  $S < S'$ . In fact, the SSEP( $K_Q^N$ ) and the SSEP( $K_N$ ) differ only by the total spin associated with each vertex set, barring the additional degeneracies induced by the permutational equivalence of states within each vertex set that we briefly discussed in Sec. IV.

### B. Concatenated bipartite graphs in a chain

If we concatenate  $N$  bipartite graphs, we obtain a graph like the one depicted partly in Fig. 3. We shall denote this graph as  $K_{Q_1, Q_2} \times K_{Q_2, Q_3} \times \dots \times K_{Q_N, Q_{N+1}}$ . Under periodic boundary conditions there is an additional link between  $K_{Q_N, Q_{N+1}}$  and  $K_{Q_1, Q_2}$ , and in this case we must have  $Q_{N+1} = Q_1$ , otherwise it is an open chain. For this graph, under open boundary conditions  $\mathcal{H}$  reads

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^N Q_i Q_{i+1} - 2 \sum_{i=1}^N \vec{S}_i \cdot \vec{S}_{i+1}, \quad (24)$$

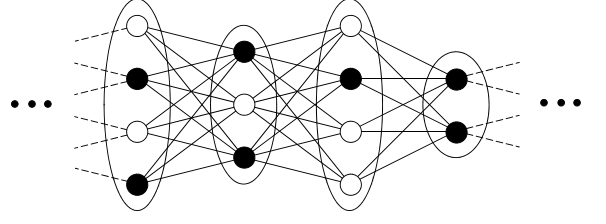


FIG. 3. Concatenated bipartite graphs in a chain. The segment shown depicts graphs  $K_{43}$ ,  $K_{34}$ , and  $K_{42}$  linked together such that every vertex is linked to every other vertex in the “neighboring” vertices sets. In general, the edges set of the composite graph under open boundary condition is given by  $E_{\text{open}} = (V_{Q_1} \times V_{Q_2}) \cup (V_{Q_2} \times V_{Q_3}) \cup \dots \cup (V_{Q_N} \times V_{Q_{N+1}})$ ; under closed boundary condition the edges set becomes  $E_{\text{closed}} = E_{\text{open}} \cup (V_{Q_{N+1}} \times V_{Q_1})$ .

where each spin operator  $\vec{S}_i$  is given as in Eq. (18). Operator (24) is equivalent to the Hamiltonian of a one-dimensional ferromagnetic Heisenberg model of mixed spins  $S_1 = Q_1/2, \dots, S_{N+1} = Q_{N+1}/2$ , with each bipartite graph  $K_{Q_i, Q_{i+1}}$  of the chain corresponding to a “unit cell.”

If all  $Q_i$  are equal, then a simple spin-wave analysis shows that the low-lying eigenspectrum just above the stationary state has the form  $E_N \propto 2 \sin^2(\pi/N)$ , with an asymptotic behavior  $E_N \sim 2\pi^2 N^{-2}$  for  $N \gg 1$  [14]. In this case, the relation between the relaxation time scale and the spectral gap becomes  $\tau \sim N^2$ , typical of diffusive behavior. It is well known that the SSEP displays this type of dispersion relation, where the dependence on the number of particles (and on the spin  $Q/2$ ) affects only prefactors, not the dependence on  $N^2$  [1, 2].

If some or all  $Q_i$  are different, then we have a full-fledged mixed-spins operator. It has been proved that the eigenspectrum of the mixed-spins chain (24) displays the ferromagnetic ordering of energy levels; actually, this property was first demonstrated for quantum spins chains exactly like (24) under open boundary conditions [16]. While mixed-spins Hamiltonians of interest in the theory of magnetism are usually antiferromagnetic, operator (24) is always ferromagnetic [28]. We thus expect that a modified spin-wave analysis already successful in the more complicated cases of antiferromagnetic or competing interactions [30] shall work even better in the analysis of (24). This provides an interesting avenue for further investigations.

## VI. SUMMARY AND OUTLOOK

The investigation of interacting particle systems on graphs is an active field of mathematical research [11–13, 19]. In the physics literature, however, one is almost always concerned with linear chains, hypercubes, and complete graphs, this last ones in the guise of mean-field approximations. We showed that other types of graphs are amenable to investigation with familiar techniques like quantum angular momentum algebra and basic group representation theory. The list of graphs that can be explored in this way includes star graphs (the star graph  $S_N$  is just the complete bipartite graph  $K_{1,N}$ ), wheel graphs, and finite regular trees; see Fig. 4.

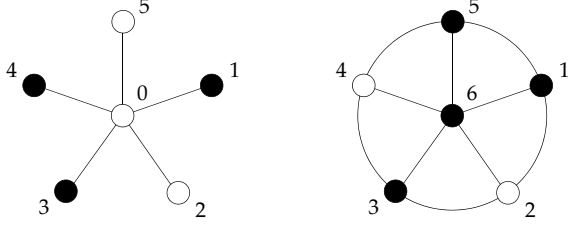


FIG. 4. Left: The star graph  $S_5$ . Star graphs  $S_N$  are equivalent to complete bipartite graphs  $K_{1,N}$ . Right: The wheel graph  $W_6$ .

We avoided employing  $SU(N)$  representations and Young tableaux on purpose to keep the exposition elementary. Indeed, the study of permutation invariant operators like (8) and (20) is more natural by means of the permutation group; see, e.g., [20] for background and [21–23] for closely related applications. Otherwise, it is clear that interacting spin waves, variational states (including matrix product ansätze), and cluster approximations, among other approaches, could also be applied in the investigation of exclusion processes on graphs, e.g., to estimate the gap of the mixed-spins operator (24).

While several works have investigated the quantum Curie-Weiss model and related XY and Ising models in transverse fields in the context of quantum phase transitions and quantum information and computation [24–27], very little, if any, has been done on the Heisenberg model on complete bipartite and multipartite graphs. These models deserve a closer look, because they are very simple and because complete bipartite and multipartite graphs appear in the discussion of several combinatorial problems that may become relevant in the field of quantum information and computation.

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### Appendix: Outer multiplicities in the Clebsch-Gordan series for arbitrary spin- $s$ representations of the $SU(2)$

The degeneracies of the eigenvalues of the operators (8) and (20) partly come from the existence of many possible different combinations of the elementary spins summing up to a definite value of  $S$ . This degeneracy is encoded in the outer multiplicity  $d_s(N, S)$  of the irreducible representation  $\mathcal{D}^{(S)}$  appearing in the  $N$ -fold tensor product

$$[\mathcal{D}^{(s)}]^{\otimes N} = \bigoplus_{S=S_{\min}}^{sN} d_s(N, S) \mathcal{D}^{(S)} \quad (\text{A.1})$$

resolved into the direct sum via repeated application of the Clebsch-Gordan series [20]

$$\mathcal{D}^{(\ell)} \otimes \mathcal{D}^{(\ell')} = \mathcal{D}^{(|\ell' - \ell|)} \oplus \mathcal{D}^{(|\ell' - \ell| + 1)} \oplus \dots \oplus \mathcal{D}^{(\ell' + \ell)}, \quad (\text{A.2})$$

where  $S_{\min} = 1/2$  if  $s$  is half-integer and  $N$  is odd and  $S_{\min} = 0$  otherwise,  $N \geq 2$  is the number of vertices of the graph, and  $s$  is the magnitude of the spins involved.

To calculate  $d_s(N, S)$ , we first notice that in a system of  $N$  spins  $s$ , the number  $b_s(N, M)$  of states of total magnetization  $M$  is given by the coefficient of  $z^M$  in the expansion of  $(z^{-s} + z^{-s+1} + \dots + z^s)^N$ . This recipe stems from the solution of the simple combinatorial problem of distributing  $M$  things among  $N$  boxes each supporting a minimum of  $-s$  and a maximum of  $+s$  things [31]. Next we notice that  $d_s(N, S)$  and the numbers  $b_s(N, M)$  are related by

$$d_s(N, S) = b_s(N, S) - b_s(N, S + 1), \quad (\text{A.3})$$

since in a given subspace of fixed  $M$  we have  $S \geq |M|$ , so that  $b_s(N, S) - b_s(N, S + 1)$  counts just those states with exactly total spin  $S$ . Finally, an explicit expression for  $b_s(N, S)$  can be obtained from its defining expansion,

$$\begin{aligned} (z^{-s} + z^{-s+1} + \dots + z^s)^N &= \sum_{M=-sN}^{+sN} b_s(N, M) z^M = \\ &= z^{-sN} (1 + z + \dots + z^{2s})^N = z^{-sN} \sum_{M=0}^{2sN} c_{2s+1}(N, M) z^M, \end{aligned} \quad (\text{A.4})$$

such that  $b_s(N, M) = c_{2s+1}(N, sN + M)$ . It is clear from Eq. (A.4) that  $b_s(N, M) = b_s(N, -M)$ . The coefficients  $c_{2s+1}(\cdot, \cdot)$  are known as generalized (or extended) binomial coefficients of order  $2s + 1$ , and reduce to the standard binomial coefficients when  $s = \frac{1}{2}$  [31, 32],

$$b_{1/2}(N, M) = c_2(N, \frac{1}{2}N + M) = \binom{N}{\frac{1}{2}N + M}; \quad (\text{A.5})$$

compare Eqs. (A.3) and (A.5) with Eq. (12). If we put  $z = 1$  in Eq. (A.4) we obtain  $\sum_M b_s(N, M) = (2s + 1)^N$ , as required.

It turns out that generalized binomial coefficients can be written in terms of standard binomial coefficients [31, 32]. The resultant expression for  $b_s(N, M)$  is

$$b_s(N, M) = \sum_{k \geq 0} (-1)^k \binom{N}{k} \binom{(s+1)N + M - (2s+1)k - 1}{sN + M - (2s+1)k}, \quad (\text{A.6})$$

where the summation runs over  $k$  as long as the summing terms are non-null. Both the upper and the lower terms in the second binomial coefficient above are integer, even if  $N$  is odd and  $s$  is half-integer, because then  $M$  will necessarily be half-integer. Equations (A.3), (A.4), and (A.6) together solve the Clebsch-Gordan series decomposition (A.1) for arbitrary spin- $s$  representations of the  $SU(2)$ . For example, we find that the irreducible representation  $\mathcal{D}^{(29/2)}$  appears  $d_{3/2}(15, 29/2) = 167895$  times in the Clebsch-Gordan series for  $[\mathcal{D}^{(3/2)}]^{\otimes 15}$  after computing just 10 terms in Eq. (A.6) for each coefficient  $b_{3/2}(15, 29/2)$  and  $b_{3/2}(15, 31/2)$  of Eq. (A.3), a modest effort for an otherwise laborious calculation. In fact, it is possible to combine (by means of elementary binomial identities) the two coefficients in Eq. (A.3) into a single factor, further reducing the calculational effort. Of course, in practice  $d_s(N, S)$  is calculated on a computer, except perhaps in small cases (say, for  $s \leq 1$  and  $N \leq 10$ ).

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